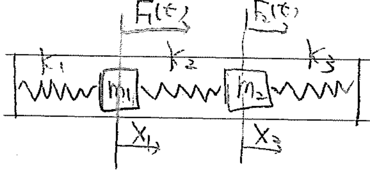
are also solutions to [a]. We find that $\{a_1 e^{\lambda_1 t}\cos\mu_1 t - b_1 e^{\lambda_1 t}\sin\mu_1 t,$ $b_1 e^{\lambda_1 t}\cos\mu_1 t + a_1 e^{\lambda_1 t}\sin\mu_1 t, \cdots, a_k e^{\lambda_k t}\cos\mu_k t - b_k e^{\lambda_k t}\sin\mu_k t, b_k e^{\lambda_k t}\cos\mu_k t$ $+ a_k e^{\lambda_k t}\sin\mu_k t, u_{2k+1} e^{\lambda_{2k+1} t}, \cdots, u_n e^{\lambda_n t}\} = \{\varphi_1(t), \cdots, \varphi_n(t)\}$ is a collection of real-valued solutions to [a]. Claim that $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of [a]. Proof of the claim: We need to prove that $W(\varphi_1, \cdots, \varphi_n)(t) \neq 0$, for all $t \in \mathbb{R}$. Let $t=0$. Then, $W(\varphi_1, \cdots, \varphi_n)(0) = \det(\varphi_1(0), \cdots, \varphi_n(0)) = \det(a_1, b_1, \cdots, a_k, b_k,$ $u_{2k+1}, \cdots, u_n)$. Check that $\{a_1, b_1, \cdots, a_k, b_k, u_{2k+1}, \cdots, u_n\}$ are linearly independent. If not, we can find $(c_1, d_1, c_2, d_2, \cdots, c_k, d_k, c_{2k+1},$ $c_{2k+2}, \cdots, c_n) \neq 0$ such that $c_1 a_1 + d_1 b_1 + \cdots + c_k a_k + d_k b_k +$ $c_{2k+1} u_{2k+1} + \cdots + c_n u_n = 0 \Rightarrow \frac{1}{2}(c_1 - i d_1)(a_1 + i b_1) + \frac{1}{2}(c_1 + i d_1)(a_1 - i b_1)$ $+ \frac{1}{2}(c_2 - i d_2)(a_2 + i b_2) + \frac{1}{2}(c_2 + i d_2)(a_2 - i b_2) + \cdots + \frac{1}{2}(c_k - i d_k)(a_k + i b_k)$ $+ \frac{1}{2}(c_k + i d_k)(a_k - i b_k) + c_{2k+1} u_{2k+1} + \cdots + c_n u_n = 0 \Rightarrow c_1 = d_1 = \cdots = c_k = d_k$ $= c_{2k+1} = \cdots = c_n = 0$. Hence $\{a_1, b_1, \cdots, a_k, b_k, u_{2k+1}, \cdots, u_n\}$ are linear independent. Hence $W(\varphi_1,\cdots,\varphi_n)(0)\neq 0$. By Thm 7.12, we deduce that $W(\varphi_1,\cdots,\varphi_n)(t)\neq 0$, for every $t\in\mathbb{R}$. Thus, $\{\varphi_1,\cdots,\varphi_n\}$ is a real-valued fundamental set of [a]. Hence the general solution of $x'=Ax$ is of the form $x(t)=\sum_{j=1}^{k}\left[C_1^{(j)}(a_j\cos\mu_j t-b_j\sin\mu_j t)+C_2^{(j)}(a_j\sin\mu_j t+b_j\cos\mu_j t)\right]e^{\lambda_j t}+\sum_{j=2k+1}^{n}C_j u_j e^{\lambda_j t}$.

Example 7.17: Consider the two-mass three-spring system

$$\begin{cases} m_1 x_1''(t)=-(k_1+k_2)x_1+k_2x_2+F_1(t) \\ m_2 x_2''(t)=k_2x_1-(k_2+k_3)x_2+F_2(t) \end{cases} \quad [*],$$

which is used to model the motion of two objects shown in the figure below:



k_1, k_2, k_3: Hooke constants

Let $y_1=x_1$, $y_2=x_2$, $y_3=x_1'$, $y_4=x_2'$. Then $y=(y_1,y_2,y_3,y_4)^T$ satisfies

$$y'=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-\frac{k_1+k_2}{m_1}&\frac{k_2}{m_1}&0&0\\\frac{k_2}{m_2}&-\frac{k_2+k_3}{m_2}&0&0\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{F_1}{m_1}\\\frac{F_2}{m_2}\end{pmatrix}.$$

Suppose $F_1(t)=0$, $F_2(t)=0$, $m_1=2$, $m_2=\frac{9}{4}$, $k_1=1$, $k_2=3$, $k_3=\frac{15}{4}$.

$\Rightarrow y'=Ay$, $A=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-2&\frac{3}{2}&0&0\\\frac{4}{3}&-3&0&0\end{pmatrix}$. The eigenvalue r of A satisfies $\det(A-rI)=0$. $\det(A-rI)=\det\begin{pmatrix}-r&0&1&0\\0&-r&0&1\\-2&\frac{3}{2}&-r&0\\\frac{4}{3}&-3&0&-r\end{pmatrix}=r^4+5r^2+4$.

$\Rightarrow \pm i, \pm 2i$ are eigenvalues of A. Let $r_1=i$, $r_2=-i$, $r_3=2i$, $r_4=-2i$.

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Corresponding eigenvectors can be chosen as $u_1 = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}$, $u_3 = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 6 \\ -8 \end{pmatrix}$, $u_4 = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 0 \\ 6 \\ -8 \end{pmatrix}$. Let $a = (3,2,0,0)^T$, $b = (0,0,3,2)^T$, $c = (3,-4,0,0)^T$, $d = (0,0,6,-8)^T$. The general solution to $y' = Ay$ is $y(t) = C_1(a\cos t - b\sin t) + C_2(a\sin t + b\cos t) + C_3(c\cos 2t - d\sin 2t) + C_4(c\sin 2t + d\cos 2t)$. $\boxed{////}$

§7.3.3. The case that A has repeated eigenvalues.

When A is diagonalizable, the discussion is pretty much the same as Sections §6.3.1 and §6.3.2. Assume A is diagonalizable, then $A = P\Lambda P^{-1}$ for some diagonal matrix Λ. Then, $x' = Ax$ with initial data $x(t_0) = x_0$ can be written as $x(t) = P e^{(t-t_0)\Lambda} P^{-1} x_0$. So we focus on the case when A is an $n\times n$ matrix which is not diagonalizable.

Recall some linear algebra: Write $\det(A - \lambda I) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2} \cdots (\lambda-\lambda_k)^{m_k}$, $m_1 + \cdots + m_k = n$. For $j = 1, \cdots, k$, let $E_{\lambda_j} := \{v \in \mathbb{C}^n \mid (A - \lambda_j I)v = 0\}$.

A is diagonalizable if and only if $\dim E_{\lambda_j} = m_j$, for every $j=1,2,\dots,k$. A is not diagonalizable if and only if there is a $j_0 \in \{1,\dots,k\}$, such that $\dim E_{\lambda_{j_0}} < m_{j_0}$.

Example 7.18: Let $A=\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}$, $\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & -1\\ 1 & 3-\lambda\end{pmatrix}=$ $(1-\lambda)(3-\lambda)+1=\lambda^2-4\lambda+4=(\lambda-2)^2$. $E_2=\{v\in\mathbb{C}^2 \mid Av=2v\}=\operatorname{span}((1,-1)^T)$.

$\Rightarrow \dim E_2 = 1 < 2 \Rightarrow$ A is not diagonalizable. $\begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$\Rightarrow \begin{cases} x_1' = x_1 - x_2 \\ x_2' = x_1 + 3x_2 \end{cases}$ [*] $\Rightarrow x_2 = x_1 - x_1'$, $\Rightarrow x_1' - x_1'' = x_1 + 3(x_1 - x_1') = 4x_1 - 3x_1'$.

$\Rightarrow x_1'' - 4x_1' + 4x_1 = 0$. This is a second order O.D.E. and the characteristic equation $r^2-4r+4=0$ has double root $r=2$.

$\Rightarrow x_1(t) = c_1 e^{2t} + c_2 t e^{2t} \Rightarrow X = (x_1(t), x_2(t))^T = (c_1 e^{2t} + c_2 t e^{2t}, -(c_1+c_2)e^{2t} - c_2 t e^{2t})^T$.

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Given an large non-diagonalizable matrix A, it is almost impossible to carry out the same computation as in Example 7.18. ∎

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Thm 7.19: Let $A \in M_{n\times n}$ be a real constant matrix. Then the solution to $x' = Ax$ with initial data $x(t_0) = x_0$ is given by $x(t) = e^{(t-t_0)A}x_0$. ▨

pf: Let $y(t) = e^{(t-t_0)A}x_0$. Then, $y(t) = \left(I + (t-t_0)A + \frac{(t-t_0)^2}{2!}A^2 + \cdots\right)x_0$ $= x_0 + (t-t_0)Ax_0 + \frac{(t-t_0)^2}{2!}A^2x_0 + \cdots + \frac{(t-t_0)^k}{k!}A^kx_0 + \cdots$. Therefore, $y'(t) = Ax_0 + (t-t_0)A^2x_0 + \cdots + \frac{(t-t_0)^{k-1}}{(k-1)!}A^kx_0 + \cdots = A\left(I + (t-t_0)A + \cdots + \frac{(t-t_0)^{k-1}}{(k-1)!}A^{k-1} + \cdots\right)x_0 = Ae^{(t-t_0)A}x_0 = Ay(t). \Rightarrow y' = Ay.$ ▨

When A is diagonalizable, $A = P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}P^{-1}$, $e^{At} = P\begin{pmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}P^{-1}$.

Q: How to compute e^{At} when A is not diagonalizable?

Example 7.20: Let $A = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$. $\det(A-\lambda I) = (\lambda-2)^2$. $E_2 = \{v \in \mathbb{C}^2 \mid Av = 2v\}$. $\dim E_2 = 1 < 2 \Rightarrow A$ is not diagonalizable. For every $k \in \mathbb{N}$, $t^kA^k = t^k\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^k$. $\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^2 = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}4 & 4\\ 0 & 4\end{pmatrix}$

$\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^3 = \begin{pmatrix}4 & 4\\ 0 & 4\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}2^3 & 3\times 2^2\\ 0 & 2^3\end{pmatrix}$, $\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^4 = \begin{pmatrix}2^3 & 12\\ 0 & 2^3\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}2^4 & 32\\ 0 & 2^4\end{pmatrix}$ $= \begin{pmatrix}2^4 & 4\times 2^3\\ 0 & 2^4\end{pmatrix}$, $\cdots$ $\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^k = \begin{pmatrix}2^k & k\cdot 2^{k-1}\\ 0 & 2^k\end{pmatrix} \Rightarrow t^kA^k = \begin{pmatrix}t^k2^k & t^kk2^{k-1}\\ 0 & t^kA^k\end{pmatrix}$.

$\Rightarrow e^{tA} = I + tA + \frac{1}{2!}(tA)^2 + \cdots + \frac{1}{k!}(tA)^k = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} & \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!}2^{k-1} \\ 0 & \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \end{pmatrix}$

$= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$. Similarly, for $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$. ▨

Def 7.21: A matrix $A \in M_{n\times n}$ is said to be of Jordan canonical form if $A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & 0 \\ 0 & \cdots & 0 & A_\ell \end{pmatrix}$, where each 0 is zero matrix and each A_j is a square matrix of the form (λ) or $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$, for some eigenvalue λ of A. ▨

Note that the diagonal elements of different A_j might be the same and a diagonal matrix is of Jordan canonical form.

Thm 7.22: Let $A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_\ell \end{pmatrix}$ be a Jordan canonical form. Then, $A^k = \begin{pmatrix} A_1^k & & 0 \\ & \ddots & \\ 0 & & A_\ell^k \end{pmatrix}$, for every $k \in \mathbb{N}$, and $e^{tA} = \begin{pmatrix} e^{tA_1} & & 0 \\ & \ddots & \\ 0 & & e^{tA_\ell} \end{pmatrix}$. ▨

pf: By some straightforward computation. ▨

Example 7.23: Let $\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Then Λ is of Jordan canonical form and $e^{t\Lambda} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$. ▨

Example 7.24: Let $\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Then Λ is of Jordan canonical

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form and $\Lambda^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}$, for every $k \in \mathbb{N}$, and $e^{t\Lambda}$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k & \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^k \lambda^{k-1} & \sum_{k=2}^{\infty} \frac{1}{2(k-2)!} t^k \lambda^{k-2} \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k & \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^k \lambda^{k-1} \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda^k \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

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Thm 7.25 Let $\Lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ is a $m \times m$ matrix. For every $k \in \mathbb{N}$, we have $\Lambda^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & C_2^k \lambda^{k-2} & \cdots & C_{m-1}^k \lambda^{k-m+1} \\ & \lambda^k & k\lambda^{k-1} & \ddots & \\ & & \ddots & \ddots & k\lambda^{k-1} \\ 0 & & & & \lambda^k \end{pmatrix}$ ⋯ [*], where $C_m^k = \frac{k!}{m!(k-m)!}$ if $k \geq m$ and $C_m^k = 0$ if $k < m$.

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pf: [*] holds for $k=1$. Assume that [*] holds for every $k \leq k_0$, $k_0 \in \mathbb{N}$. We are going to prove that [*] holds for $k = k_0+1$.

$$\Lambda^{k_0+1} = \Lambda^{k_0}\Lambda = \begin{pmatrix} \lambda^{k_0} & k_0\lambda^{k_0-1} & C_2^{k_0}\lambda^{k_0-2} & \cdots & C_{m-1}^{k_0}\lambda^{k_0-m+1} \\ & \lambda^{k_0} & \ddots & & \\ & & \ddots & & k_0\lambda^{k_0-1} \\ & & & & \lambda^{k_0} \end{pmatrix} \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \\ & & & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{k_0+1} & \lambda^{k_0} + k\lambda^{k_0}, & k_0\lambda^{k_0-1} + \lambda C_2^{k_0}\lambda^{k_0-2}, & C_2^{k_0}\lambda^{k_0-2} + C_3^{k_0}\lambda^{k_0-2}, & \cdots, & C_j^{k_0}\lambda^{k_0-j} + C_{j+1}^{k_0}\lambda^{k_0-j}, & \cdots, & C_{m-2}^{k_0}\lambda^{k_0-m+2} + C_{m-1}^{k_0} \\ 0 & \lambda^{k_0+1} & & & & & & \\ & & \ddots & & & & & \\ 0 & & & & & & & \end{pmatrix}$$

$\lambda^{k_0} + k\lambda^{k_0} = (k+1)\lambda^{k_0}$, $k_0\lambda^{k_0-1} + \lambda C_2^{k_0}\lambda^{k_0-2} = \lambda^{k_0-1}(k_0 + C_2^{k_0}) = \lambda^{k_0-1}(C_1^{k_0} + C_2^{k_0})$

$= \lambda^{k_0-1} C_2^{k_0+1}$. $C_j^{k_0}\lambda^{k_0-j} + C_{j+1}^{k_0}\lambda^{k_0-j} = \lambda^{k_0-j}(C_j^{k_0} + C_{j+1}^{k_0}) = \lambda^{k_0-j} C_{j+1}^{k_0+1}$.

$\Rightarrow$ [*] holds for $k = k_0+1$. By induction, the theorem follows.

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Let $A = \begin{pmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}$ be a $m \times m$ matrix. From Thm 7.25, We deduce that $e^{tA} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2!}t^2 e^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ & e^{\lambda t} & \ddots & & \vdots \\ & & \ddots & & te^{\lambda t} \\ & & & & e^{\lambda t} \end{pmatrix}$ … [*I]

Thm 7.26: Let $A \in M_{n\times n}$. There exists an invertible $n\times n$ matrix P and a matrix Λ of Jordan canonical form such that $A = P\Lambda P^{-1}$.

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Given a Jordan decomposition $A = P\Lambda P^{-1}$, we have $e^{tA} = Pe^{t\Lambda}P^{-1}$. We can compute $e^{t\Lambda}$ by using Thm 7.22 and [*I]. Thus the computation of e^{tA} becomes easier as long as we know how to find the decomposition $A = P\Lambda P^{-1}$.

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Lecture XIII. (6月4日)

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Review some linear algebra: How to obtain a Jordan decomposition of a matrix $A \in M_{n\times n}$.

Def 7.27: Let $A \in M_{n\times n}$. A vector $v \in \mathbb{C}^n$ is called a generalized eigenvector of A associated with λ if $(A-\lambda I)^p v = 0$ for some positive $p \in \mathbb{N}$. $\boxed{///}$

If v is a generalized eigenvector of A associated with λ and p is the smallest positive integer for which $(A-\lambda I)^p v = 0$. Then, $(A-\lambda I)^{p-1} v \neq 0$ and $(A-\lambda I)^{p-1} v$ is an eigenvector of A associated with λ. Therefore λ is an eigenvalue of A.

Def 7.28 (Generalized eigenspaces): Let $A \in M_{n\times n}$ and λ be an eigenvalue of A. The generalized eigenspace of A associated with λ, denoted by K_λ, is the subspace of $\mathbb{C}^n$ given by

$K_\lambda := \{ v \in \mathbb{C}^n \mid (A-\lambda I)^p v = 0, \text{ for some positive integer } p \}$. $\boxed{///}$

The construction of Jordan decomposition:

For simplicity, we assume that $\det(A-\lambda I) = (\lambda-\lambda_0)^n$. We look for an invertible matrix P so that $A = P\Lambda P^{-1}$, Λ: Jordan form. Let $E_{\lambda_0} = \{v \in \mathbb{C}^n \mid (A-\lambda_0 I)v = 0\}$, $K_{\lambda_0} = \{v \in \mathbb{C}^n \mid (A-\lambda_0)^p v = 0, \text{ for some } p \in \mathbb{N}\}$. Let $r = \dim E_{\lambda_0}$.

It is well-known that (linear algebra), $\dim K_{\lambda_0} = n$. ① Determine the smallest integer k such that $n = \dim(\operatorname{Ker}(A-\lambda_0 I)^k)$. For simplicity we assume that $k=3$. ② Find $p_\ell = \dim(\operatorname{Ker}(A-\lambda_0 I)^\ell)$, $\ell = 1,2,3$.

Note that $0 < p_1 < p_2 < p_3 = n$. Let $\{v_1, v_2, \cdots, v_{s_1}\}$ be a basis for $\operatorname{Ker}(A-\lambda_0 I)^3 / \operatorname{Ker}(A-\lambda_0 I)^2$.

Let $\{(A-\lambda_0 I)v_1, \cdots, (A-\lambda_0 I)v_{s_1}, v_{s_1+1}, \cdots, v_{s_1+s_2}\}$ be a basis for $\operatorname{Ker}(A-\lambda_0 I)^2 / \operatorname{Ker}(A-\lambda_0 I)^1$.

Let $\{(A-\lambda_0 I)^2 v_1, \cdots, (A-\lambda_0 I)^2 v_{s_1}, (A-\lambda_0 I)v_{s_1+1}, (A-\lambda_0 I)v_{s_1+2}, \cdots, (A-\lambda_0 I)v_{s_1+s_2}, v_{s_1+s_2+1}, \cdots, v_{s_1+s_2+s_3}\}$ be a basis for $\operatorname{Ker}(A-\lambda_0 I)$.

Note that $s_1 = p_3 - p_2$, $s_1+s_2 = p_2 - p_1$, $s_1+s_2+s_3 = p_1$. Let $W_1 = \{\underbrace{v_{s_1+s_2+1}}_{\alpha_1}, \cdots, \underbrace{v_{s_1+s_2+s_3}}_{\alpha_{s_1+s_2+s_3}}\}$, $W_2 = \{\underbrace{(A-\lambda_0 I)v_{s_1+1}}_{\beta_1}, \underbrace{v_{s_1+1}}_{\gamma_1}, \underbrace{(A-\lambda_0 I)v_{s_2+1}}_{\beta_2}, \underbrace{v_{s_2+1}}_{\gamma_2}, \cdots, \underbrace{(A-\lambda_0 I)v_{s_1+s_2}}_{\beta_{s_1+s_2}}, \underbrace{v_{s_1+s_2}}_{\gamma_{s_1+s_2}}\}$; $W_3 = \{\underbrace{(A-\lambda_0 I)^2 v_1}_{a_1}, \underbrace{(A-\lambda_0 I)v_1}_{b_1}, \underbrace{v_1}_{c_1}, \underbrace{(A-\lambda_0 I)^2 v_2}_{a_2}, \underbrace{(A-\lambda_0 I)v_2}_{b_2}, \underbrace{v_2}_{c_2}, \cdots, \underbrace{(A-\lambda_0 I)^2 v_{s_1}}_{a_{s_1}}, \underbrace{(A-\lambda_0 I)v_{s_1}}_{b_{s_1}}, \underbrace{v_{s_1}}_{c_{s_1}}\}$. Then $\{\alpha_1, \cdots, \alpha_{s_1+s_2+s_3}, \beta_1, \gamma_1, \cdots, \beta_{s_1+s_2}, \gamma_{s_1+s_2}, a_1, b_1, c_1, \cdots, a_{s_1}, b_{s_1}, c_{s_1}\}$ is a basis for $\mathbb{C}^n$.

Now $A\alpha_1=\lambda_0\alpha_1, \cdots, A\alpha_{s_1+s_2+s_3}=\lambda_0\alpha_{s_1+s_2+s_3}$, $A\beta_1=\lambda_0\beta_1$, $A\gamma_1=\beta_1+\lambda_0\gamma_1=\underbrace{(A-\lambda_0 I)\gamma_1}_{=\beta_1}+\lambda_0\gamma_1$, $\left(\begin{pmatrix}\lambda_0 & 1\\ 0 & \lambda_0\end{pmatrix}\rightarrow\text{with respece to }\{\beta_1,\gamma_1\}.\right)$

$\cdots$, $A\beta_{s_1+s_2}=\lambda_0\beta_{s_1+s_2}$, $A\gamma_{s_1+s_2}=\beta_{s_1+s_2}+\lambda_0\gamma_{s_1+s_2}$, $Aa_1=\lambda_0 a_1$,

$Ab_1=a_1+\lambda_0 b_1$, $Ac_1=(A-\lambda_0 I)c_1+\lambda_0 c_1=b_1+\lambda_0 c_1$. $\left(\begin{pmatrix}\lambda_0 & 1 & 0\\ 0 & \lambda_0 & 1\\ 0 & 0 & \lambda_0\end{pmatrix}\rightarrow\text{with respect to }\{a_1,b_1,c_1\}.\right)$, $\cdots$, $Aa_{s_1}=\lambda_0 a_{s_1}$, $Ab_{s_1}=a_{s_1}+\lambda_0 b_{s_1}$, $Ac_{s_1}=b_{s_1}+\lambda_0 c_{s_1}$. We have obtained a Jordan decomposition for A ////

Example 7.29: Find the Jordan decomposition of the matrix

$A=\begin{pmatrix}4&-2&0&2\\0&6&-2&0\\0&2&2&0\\0&-2&0&6\end{pmatrix}$. $\det(A-\lambda I)=\det\begin{pmatrix}4-\lambda&-2&0&2\\0&6-\lambda&-2&0\\0&2&2-\lambda&0\\0&-2&0&6-\lambda\end{pmatrix}=(\lambda-4)^3(\lambda-6)$.

We can check that $\dim(\operatorname{Ker}(A-4I))=2$, $\dim(\operatorname{Ker}(A-4I)^2)=3$. $\operatorname{Ker}(A-4I)=\operatorname{span}((1,0,0,0)^T,(0,1,1,1)^T)$, $\operatorname{Ker}(A-4I)^2=\operatorname{span}((1,0,0,0)^T,(0,1,0,2)^T,(0,1,2,0)^T)$. Note that either $(0,1,0,2)^T$ or $(0,1,2,0)^T$ is in $\operatorname{Ker}((A-4I))$. We can take $v=(0,1,0,2)^T$ as a basis for the quotient space $\operatorname{Ker}(A-4I)^2/\operatorname{Ker}(A-4I)$ $\Rightarrow (A-4I)v=(2,2,2,2)^T$. Take $(1,0,0,0)^T$ so that $\{(1,0,0,0)^T, (0,1,0,2)^T, (A-4I)(0,1,0,2)^T = (2,2,2,2)^T\}$ is a basis for $\operatorname{Ker}((A-4I)^2)$. Moreover, $(1,0,0,1)^T$ is an eigenvector of A associated to $\lambda = 6$. Let $P = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix}$, with columns u_1, u_2, u_3, u_4. Then $A = P \Lambda P^{-1}$,

$\Lambda = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$.

$\blacksquare$

<u>Example 7.30</u>: Let $A = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & 0 & 6 \end{pmatrix}$. Find general solution to $x' = Ax$. The general solution is given by $x(t) = e^{tA} x_0$, for some vector $x_0 \in \mathbb{R}^4$. Now, $e^{tA} = P e^{t\Lambda} P^{-1}$. By [$*$I], we have $e^{t\Lambda} = \begin{pmatrix} e^{4t} & te^{4t} & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \Rightarrow x(t) = (u_1, u_2, u_3, u_4) \begin{pmatrix} e^{4t} & te^{4t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$, $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = P^{-1} x_0 \Rightarrow x(t) = (u_1, u_2, u_3, u_4) \begin{pmatrix} c_1 e^{4t} + c_2 te^{4t} \\ c_2 e^{4t} \\ c_3 e^{4t} \\ c_4 e^{6t} \end{pmatrix} = (c_1 e^{4t} + c_2 te^{4t}) u_1 + c_2 e^{4t} u_2 + c_3 e^{4t} u_3 + c_4 e^{6t} u_4$.

$\blacksquare$

§7.4: Fundamental matrices: Consider $x' = P(t)x$, [$*$] $P(t): I \to M_{n\times n}$ is a matrixed valued function. Let $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ be a fundamental set of [$*$]. Recall that (see Def 7.11), the fundamental matrix for [$*$] is the matrix $\Phi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$.

V. ✻✻

Lecture XIII.

Since each column of Ψ is a solution to [✻], we have that $\Psi'(t) = P(t)\Psi(t) \Rightarrow \Psi'(t)\Psi(t)^{-1} = P(t)$, for all $t \in I$. A special kind of fundamental matrix Φ whose initial value $\Phi(t_0)$ is the identity matrix, is in particular helpful for constructing solutions to $\begin{cases} x' = P(t)x \\ x(t_0) = x_0 \end{cases}$. Let $x(t) = \Phi(t)x_0$. Then, $x(t_0) = \Phi(t_0)x_0 = x_0$ and $x'(t) = \Phi'(t)x_0 = P(t)x_0$.

Property: For each fundamental matrix Ψ of [✻], we have

$\Psi(t) = \Phi(t)\Psi(t_0)$. ... [i]

$\boxed{///}$

pf: Let $\Psi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$, where $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ is a fundamental set for [✻]. Moreover, $\varphi_j'(t) = P(t)\varphi_j$, $j = 1, \cdots, n$. Now, consider $\hat{\varphi}_j(t) := \Phi(t)\varphi_j(t_0)$, $j = 1, \cdots, n$. For every $j = 1, \cdots, n$, we have $\hat{\varphi}_j'(t) = P(t)\hat{\varphi}_j(t)$ and $\hat{\varphi}_j(t_0) = \Phi(t_0)\varphi_j(t_0)$. By Thm 7.7, $\hat{\varphi}_j(t) = \varphi_j(t)$. Hence, $\Phi(t)\varphi_j(t_0) = \varphi_j(t)$, $j = 1, \cdots, n$. Thus, $\Phi(t)\Psi(t_0) = \Psi(t)$.

$\boxed{///}$

By [i], given a fundamental matrix Ψ, we can easily construct the fundamental matrix $\Phi(t)$ by $\Phi(t) = \Psi(t)\Psi(t_0)^{-1}$. $\boxed{///}$

§7.5: Non-homogeneous linear system. Consider the non-homogeneous

$$\begin{cases} x' = P(t)x + g(t); \\ x(t_0) = x_0 \end{cases} \quad [a] \quad \text{for some non-zero vector-valued function } g.$$

To solve [a], we need to find a particular solution $x_p(t)$. If we could find such $x_p(t)$, then the general solution to [a] is given by $x(t) = \Psi(t)C_1 + x_p(t)$, where Ψ is a fundamental matrix of $x' = P(t)x$ [b] and C_1 is an arbitrary constant vector. To satisfy the initial data [a], we let $C_1 = \Psi(t_0)^{-1}(x_0 - x_p(t_0))$ and the solution to [a] is $x(t) = \Psi(t)\Psi(t_0)^{-1}(x_0 - x_p(t_0)) + x_p(t)$.

Thm 7.31: Let $\Psi(t)$ be a fundamental matrix of system $x' = P(t)x$ and $\varphi(t)$ be the solution to the non-homogeneous system $\begin{cases} x'(t) = P(t)x + g(t) \\ x(t_0) = x_0 \end{cases}$. Then $\varphi(t) = \Psi(t)\Psi(t_0)^{-1}x_0 + \int_{t_0}^{t} \Psi(t)\Psi(s)^{-1}g(s)\,ds$ $\boxed{///}$

pf: $\varphi'(t) = \Psi'(t)\Psi(t_0)^{-1}x_0 + \Psi(t)\Psi(t)^{-1}g(t) + \int_{t_0}^{t}\Psi'(t)\Psi(s)^{-1}g(s)\,ds$

$= \Psi'(t)\Psi(t)^{-1}\left(\Psi(t)\Psi(t_0)^{-1}x_0 + \int_{t_0}^{t}\Psi(t)\Psi(s)^{-1}g(s)\,ds\right) +$

$g(t) = P(t)\varphi(t) + g(t).$ $\blacksquare$

Note that $x_p(t) = \Psi(t)\int_{t_0}^{t}\Psi(s)^{-1}g(s)\,ds$ is a particular solution to $x' = P(t)x + g(t)$. [ii]

Example 7.32: Let $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $g(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$. Find a particular solution of $x' = Ax + g(t)$. $\det(A - \lambda I) = (-2-\lambda)^2 - 1$. $\Rightarrow \lambda = -1$ and $\lambda = -3$ are eigenvalues of A. The corresponding eigenvectors are $(1, 1)^T$ and $(1, -1)^T$. Thus, $\left\{\varphi_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \varphi_2(t) = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is a fundamental set of $x' = Ax$. By (ii), we find that

$$x_p(t) = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & e^{-3s} \\ e^{-s} & -e^{-3s} \end{pmatrix}^{-1}\begin{pmatrix} 2e^{-s} \\ 3s \end{pmatrix} ds$$

is a particular solution to $x' = Ax + g(t)$. By some straightforward computation, we can check that

$$x_p(t) = \frac{1}{2}\begin{pmatrix} 2te^{-t} + 3(t-1) + e^{-t} - (t - \frac{1}{3}) \\ 2te^{-t} + 3(t-1) - e^{-t} + (t - \frac{1}{3}) \end{pmatrix}$$

$\blacksquare$